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An upper bound on the derivational complexity of Knuth–Bendix orderings

Dieter Hofbauer

Universität Kassel, Fachbereich 17 Mathematik/Informatik, D-34109 Kassel, Germany

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Abstract

The derivational complexity of a terminating rewrite system is a measure for the maximal length of rewrite sequences. We study the influence of certain standard termination criteria on the derivational complexity. In this paper we prove a uniform multiple recursive upper bound for Knuth–Bendix orderings. This continues work by Hofbauer and Lautemann [13], where it has been shown that primitive recursive bounds are impossible.

Keywords: Term rewriting; Termination proofs; Knuth–Bendix order; Derivational complexity

1. Introduction

Termination proof methods for term rewrite systems have attracted much attention in the past two decades since they are essential for a large variety of verification methods, such as confluence criteria, completion procedures, and inductive proofs. The termination criteria developed there turn out to be applicable to more general deduction processes; see Padawitz [23] for instance.

In order to investigate the power as well as the limitations of different termination proof methods, it is natural to ask how long derivation sequences can get when the termination proof is based on a certain well-founded ordering. The *derivation height* of a term t modulo some finite terminating rewrite system R is defined as the length of a longest R-derivation starting with t,

$$\mathrm{dh}_R(t) = \max\{n \in \mathbb{N} \mid \text{there is a term } s \text{ such that } t \to_R^n s\},$$

E-mail address: dieter@theory.informatik.uni-kassel.de.

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whereas the *derivational complexity* measures the worst case derivation height for terms of a given size:

$$\mathrm{dc}_R(n) = \max\{\mathrm{dh}_R(t) \mid |t| \leqslant n\}.$$

The derivational complexity of most of the standard reduction orderings has been studied. For multiset path orderings we get primitive recursive derivational complexity [10,12]. This class of termination orderings in fact characterizes primitive recursively length bounded computations on terms, even for nonconfluent systems [11]. Similarly, lexicographic path orderings yield multiple recursive derivational complexity [30].

General upper bounds can always be found in appropriate subrecursive function hierarchies; see [7,25,29] among others (these rather enormous general bounds have recently proven to be optimal [17,26–28]). For the Knuth–Bendix orderings, which are the subject of this paper, these techniques can be used to show multiple recursive complexity bounds [25]. In this way, however, one exhausts the whole hierarchy of multiple recursive functions. Here, by combinatorial means, we show the existence of a uniform upper bound in a finite initial segment of that hierarchy.

Knuth–Bendix orderings were introduced by Knuth and Bendix [14] for proving termination of rewrite systems in the context of a completion procedure. For applications of these reduction orderings see Le Chenadec [6] or Benninghofen et al. [3]. Since most of the examples in [14] are variants of certain groups, it is not surprising that Knuth–Bendix orderings are tailored to just that kind of variety. This turned out to be true even in a precise technical sense: the signature of totally free groups can be used as a standard signature for all Knuth–Bendix termination proofs, as explained in Section 4. The proof of our upper bound result is considerably simplified by this observation.

The question whether a given finite rewrite system has a termination proof via some Knuth–Bendix ordering is decidable. Algorithms for choosing the precedence and the weight function have been suggested and implemented respectively by, e.g., Lankford [16], Martin [20], Altendorf [1], Martin et al. [9].

In contrast to multiset path orderings, Knuth–Bendix orderings can yield termination proofs for rewrite systems where no primitive recursive upper bound on the derivational complexity exists. However, for the special case of monadic signatures, corresponding to string rewriting, a single exponential upper bound is obtained. The same upper bound is guaranteed for rewrite systems where no rule increases the number of "special" symbols (for groups, this is the unary inverse function). All upper bounds have been shown to be tight. After shortly reviewing these results from [13] in Sections 5 and 6, we prove the existence of a uniform multiple recursive upper bound in Section 6. This paper contains unpublished results from [10, Chap. 5].

2. Preliminaries

In this section, a few notations for first-order term rewriting are collected. For further definitions we refer to Baader and Nipkow [2] or Dershowitz and Jouannaud [8].

A signature is a finite ranked alphabet, where each symbol has a fixed arity. The set of ground (i.e., variable free) terms over signature Σ is denoted by $\mathcal{T}_\Sigma$, whereas $\mathcal{T}_\Sigma(X)$ is the set of terms allowing variables from a set X, which is disjoint to Σ. A term $t = f(t_1, \ldots, t_n)$ has the top symbol

f of arity $n \geqslant 0$ and the n-tuple of subterms $(t_1, \ldots, t_n)$, denoted by $\mathrm{top}(t)$ and $\mathrm{subterms}(t)$, respectively. For unary symbols f and $i \geqslant 0$ we inductively define the term $f^i(t)$ by $f^0(t) = t$ and $f^{i+1}(t) = f(f^i(t))$. Parentheses are often omitted for unary symbols. The number of occurrences of a symbol f (or a variable x) in t is $|t|_f$ ($|t|_x$ resp.); the number of all occurrences in t is $|t|$, the size of t. A signature is said to be monadic if each symbol has arity 1 or is a constant. This is particularly relevant for studying string rewrite systems, as strings can be naturally identified with terms over a monadic signature with one constant. Letters become unary symbols and rules are translated accordingly. E.g., the string rule $ab \to ba$ becomes the term rule $a(b(x)) \to b(a(x))$.

A term rewrite system R induces the rewrite relation $\to_R$ on $\mathcal{T}_\Sigma(X)$. We write $\to_R^n$ for the n-fold composition of $\to_R$. The system R is terminating (or Noetherian) if the transitive closure of $\to_R$ is well-founded; termination guarantees the existence of a normal form for each term t. Convergent systems are additionally confluent; here each term t has the unique normal form $t{\downarrow_R}$.

A system R is terminating if, and only if, $R \subseteq >$ for some well-founded rewrite order $>$ (i.e., a partial order which is closed under contexts and substitutions). If a rewrite order $>$ has the subterm property $f(\ldots, t_i, \ldots) > t_i$ then it is a simplification order. Simplification orders are known to be well-founded by Higman's theorem.

The (left-right) lexicographic extension of an ordering $>$ to tuples of fixed length over its domain is denoted by $>^{\mathrm{lex}}$.

3. Knuth–Bendix orderings

Knuth–Bendix orderings use a *precedence*, that is, a strict partial order $\succ$ on the underlying signature Σ, as well as a *weight function* for Σ, that is, a function $w : \Sigma \to \mathbb{N}$. The weight function is extended to a function $w : \mathcal{T}_\Sigma \to \mathbb{N}$ on ground terms by $w(f(t_1, \ldots, t_n)) = w(f) + \sum_{i=1}^n w(t_i)$. (The sum is assumed to give 0 for $n = 0$.) Terms are compared by first comparing their weights, then inspecting their top symbols with respect to the precedence, and eventually—in the case of equal weight and equal top symbols—recursively checking the list of subterms lexicographically from left to right.

Definition 3.1. The *Knuth–Bendix ordering* $\succ_{\mathrm{KB}}$ (wrt $\succ$ and w) is recursively defined as the least ordering on $\mathcal{T}_\Sigma$ such that $t \succ_{\mathrm{KB}} s$ if
- $\mathrm{w}(t) > \mathrm{w}(s)$, or
- $\mathrm{w}(t) = \mathrm{w}(s)$ and $\mathrm{top}(t) \succ \mathrm{top}(s)$, or
- $\mathrm{w}(t) = \mathrm{w}(s)$, $\mathrm{top}(t) = \mathrm{top}(s)$ and $\mathrm{subterms}(t) \succ_{\mathrm{KB}}^{\mathrm{lex}} \mathrm{subterms}(s)$.

This ordering is well founded only under additional assumptions. Consider the following examples.
1. Let c be a constant, g be a binary symbol, and define $\mathrm{w}(c) = \mathrm{w}(g) = 0$ and $c \succ g$. Then $c \succ_{\mathrm{KB}} g(c, c) \succ_{\mathrm{KB}} g(g(c, c), c) \succ_{\mathrm{KB}} \ldots$ is an infinite decreasing chain modulo $\succ_{\mathrm{KB}}$.
2. Let f be unary, $\mathrm{w}(f) = 0$, and t be some term such that $\mathrm{top}(t) \succ f$. Then we get the infinite decreasing chain $t \succ_{\mathrm{KB}} f(t) \succ_{\mathrm{KB}} f(f(t)) \succ_{\mathrm{KB}} \ldots$.

These examples are excluded, by the method of "exception-barring" [15], when we add the following two conditions.

(i) $\mathrm{w}(c) > 0$ for all constant symbols $c \in \Sigma$,
(ii) if $\mathrm{w}(f) = 0$ for some unary symbol $f \in \Sigma$ then f is a greatest symbol in Σ modulo $\succ$; i.e., $f \succ g$ for all $g \in \Sigma$, $f \neq g$.

We shall call a function symbol *special* if it is unary and has weight 0; note that condition (ii) implies that a signature contains at most one special symbol.

Usually, $\succ_{\mathrm{KB}}$ is extended to terms with variables. For that purpose we need to extend the weight function, too. Define $\mathrm{w}(x) = \min\{\mathrm{w}(c) \mid c \text{ a constant}\}$ for $x \in X$, and use the same recursive definition for w as above. Then the following definition yields a simplification ordering on $\mathcal{T}_\Sigma(X)$ under the assumptions (i) and (ii).

Definition 3.2. The *Knuth–Bendix ordering* $\succ_{\mathrm{KB}}$ (wrt $\succ$ and w) is the least ordering on $T_\Sigma(X)$ such that $t \succ_{\mathrm{KB}} s$ if $|t|_x \geqslant |s|_x$ for all $x \in X$ and

- $\mathrm{w}(t) > \mathrm{w}(s)$, or
- $\mathrm{w}(t) = \mathrm{w}(s)$ and $\mathrm{top}(t) > \mathrm{top}(s)$, or
- $\mathrm{w}(t) = \mathrm{w}(s)$, $\mathrm{top}(t) = \mathrm{top}(s)$, and $\mathrm{subterms}(t) \succ_{\mathrm{KB}}^{\mathrm{lex}} \mathrm{subterms}(s)$, or
- $t = f^i(x)$ and $s = x$ for special f, $i > 0$, $x \in X$.

Note that $|t|_x \geqslant |s|_x$ for $x \in X$ is a necessary condition for $\succ_{\mathrm{KB}}$ to be closed under substitutions. To see this, let $|s|_x > |t|_x > 0$ for some variable x, and let t' be a term such that $w(t') > w(t)$ (as $|s|_x > 1$ implies the existence of a symbol of arity greater than one, by (i) we know that terms of sufficiently large weight exist); then $s\{x \mapsto t'\} \succ_{\mathrm{KB}} t\{x \mapsto t'\}$, even in case $t \succ_{\mathrm{KB}} s$ holds true. In the other case $|s|_x > |t|_x = 0$ we get $s\{x \mapsto t\} \succeq_{\mathrm{KB}} t\{x \mapsto t\} = t$ by the subterm property (where $\succeq_{\mathrm{KB}}$ is the reflexive closure of $\succ_{\mathrm{KB}}$).

Example 3.1. Let $\Gamma = \{\mathrm{i}, \circ, \mathrm{e}\}$ be the signature of the variety of free groups, where i is unary, $\circ$ is binary, and e is a constant symbol. Define a (total) precedence $\triangleright$ on Γ by $\mathrm{i} \triangleright \circ \triangleright \mathrm{e}$ and choose $\mathrm{w}(\mathrm{i}) = \mathrm{w}(\circ) = 0$, $\mathrm{w}(\mathrm{e}) = 1$. Then, for $x, y, z \in X$,

$$\begin{aligned}
(x \circ y) \circ z &\triangleright_{\mathrm{KB}} x \circ (y \circ z), \\
\mathrm{e} \circ x &\triangleright_{\mathrm{KB}} x, \\
\mathrm{i}(x) \circ x &\triangleright_{\mathrm{KB}} \mathrm{e}, \\
\mathrm{i}(\mathrm{i}(x)) &\triangleright_{\mathrm{KB}} x,
\end{aligned}$$

as well as the conditions (i) and (ii) are easily verified. In the remainder of the paper, $\trianglerighteq_{\mathrm{KB}}$ will denote the reflexive closure of $\triangleright_{\mathrm{KB}}$.

Every Knuth–Bendix ordering is total on ground terms for total precedence, and it is isotonic (or “incremental”) in its precedence, that is, if $\succ \subseteq \succ'$ for precedences $\succ$, $\succ'$ then $\succ_{\mathrm{KB}} \subseteq \succ'_{\mathrm{KB}}$ (wrt a fixed weight function).

4. Standardizing signatures

In this section we prove a useful standardization result that will allow us to restrict proofs for upper bounds on the derivational complexity to rewrite systems over a single standard signature. It is shown that the signature Γ from Example 3.1 with the precedence and the weight function given there has the following property.

Let Σ be a signature and let $\succ$ and w be a precedence and a weight function on Σ, respectively. Then there exists a function $\kappa : \mathcal{T}_\Sigma \to \mathcal{T}_\Gamma$ such that, for $t, s \in \mathcal{T}_\Sigma$,

$$t \succ_{\mathrm{KB}} s \quad \text{implies} \quad \kappa(t) \triangleright_{\mathrm{KB}} \kappa(s).$$

As a consequence, upper bounds on the length of chains modulo $\triangleright_{\mathrm{KB}}$ for terms over Γ can be used to derive upper bounds on the length of chains modulo $\succ_{\mathrm{KB}}$ for terms over Σ. The construction is essentially the one given in [13] (see the remark below) but uses a rewrite relation for defining κ.

For the rest of this section fix Σ, $\succ$, and w such that conditions (i) and (ii) are satisfied. Without loss of generality, let $\Sigma = \{f_1, \ldots, f_m\}$, $m > 0$, and assume $\succ$ to be total: $f_m \succ \cdots \succ f_1$. Note that under these assumptions only f_m can be special. We will define κ by

$$\kappa(t) = t{\downarrow}_K$$

for a convergent rewrite system K. For this purpose we define terms over $\{\circ, \mathrm{e}\}$ that will be used to simulate precedence and weight resp. on Σ within the rules of system K. This simulation heavily depends on the ability to compare subterms lexicographically.

For simulating precedence, choose $p \in \mathbb{N}$ such that there are m different terms $p_1, \ldots, p_m$ over signature $\{\circ, \mathrm{e}\}$ with $\mathrm{w}(p_i) = p$ for $1 \leqslant i \leqslant m$; note that $p > 0$ and $|p_i| = 2p - 1$. Since $\triangleright_{\mathrm{KB}}$ on $\mathcal{T}_\Gamma$ is total, we can assume $p_m \triangleright_{\mathrm{KB}} \ldots \triangleright_{\mathrm{KB}} p_1$. For simulating weight, define a function $\omega : \Sigma \to \mathbb{Z}$ by

$$\omega(f_i) = p(2\mathrm{w}(f_i) + \mathrm{arity}(f_i) - 2).$$

For each $f_i \in \Sigma$ with $w(f_i) > 0$ choose a term w_i over signature $\{\circ, \mathrm{e}\}$ such that

$$\mathrm{w}(w_i) = \omega(f_i).$$

This is possible since for each $n > 0$ there are terms over $\{\circ, \mathrm{e}\}$ of weight n (note that $\mathrm{w}(\mathrm{e}) = 1$ and $\mathrm{w}(\mathrm{e} \circ t) = 1 + \mathrm{w}(t)$). As can easily be seen, for nonspecial f_i conditions (i) and (ii) together with $p > 0$ imply $\omega(f_i) \geqslant 0$ and

$$\omega(f_i) = 0 \quad \text{iff } \mathrm{arity}(f_i) = 0 \text{ and } \mathrm{w}(f_i) = 1, \text{ or } \mathrm{arity}(f_i) = 2 \text{ and } \mathrm{w}(f_i) = 0.$$

We are now ready to define the rewrite system K over $\Sigma \cup \Gamma$. It consists of exactly m rules according to the following schemes ($f_i \in \Sigma$ for $1 \leqslant i \leqslant m, n \geqslant 0, x_1, \ldots, x_n \in X$):

$$\begin{array}{ll}
f_m(x_1) \to \mathrm{i}(x_1) & \text{if } f_m \text{ is special } (\mathrm{arity}(f_m) = 1,\ \mathrm{w}(f_m) = 0),\\
f_i(x_1, \ldots, x_n) \to p_i \circ (x_1 \circ (\cdots \circ (x_n \circ w_i) \cdots)) & \text{if } f_i \text{ is not special}, \omega(f_i) > 0,\\
f_i \to p_i & \text{if } f_i \text{ is not special},\ \mathrm{arity}(f_i) = 0,\ \mathrm{w}(f_i) = 1,\\
f_i(x_1, x_2) \to p_i \circ (x_1 \circ x_2) & \text{if } f_i \text{ is not special},\ \mathrm{arity}(f_i) = 2,\ \mathrm{w}(f_i) = 0.
\end{array}$$

System K is convergent and for all terms $t \in \mathcal{T}_\Sigma$ we have $t{\downarrow}_K \in \mathcal{T}_\Gamma$. Lemma 4.1 relates t to $t{\downarrow}_K$, and Lemma 4.2 states the intended compatibility of κ, that is, normalization wrt K, and the Knuth–Bendix ordering.

Lemma 4.1. *Let* $t \in \mathcal{T}_\Sigma$.
1. $\mathrm{w}(t{\downarrow}_K) = p(2\mathrm{w}(t) - 1)$.
2. $|t{\downarrow}_K|_{\mathrm{i}} = |t|_{f_m}$ *if* f_m *is special*, $|t{\downarrow}_K|_{\mathrm{i}} = 0$ *else*.
3. $|t{\downarrow}_K| = O(|t|)$.

Lemma 4.2. *Let $t, s \in \mathcal{T}_\Sigma$. If $t \succ_{\mathrm{KB}} s$ then $t{\downarrow_K} \triangleright_{\mathrm{KB}} s{\downarrow_K}$.*

Proof. By induction on t.

Case 1. If $\mathrm{w}(t) > \mathrm{w}(s)$ then $\mathrm{w}(t{\downarrow_K}) = p(2\mathrm{w}(t) - 1) > p(2\mathrm{w}(s) - 1) = \mathrm{w}(s{\downarrow_K})$ by Lemma 4.1 (1), thus $t{\downarrow_K} \triangleright_{\mathrm{KB}} s{\downarrow_K}$.

Case 2. Otherwise $\mathrm{w}(t) = \mathrm{w}(s)$, thus $\mathrm{w}(t{\downarrow_K}) = \mathrm{w}(s{\downarrow_K})$ by Lemma 4.1 (1).

(2a) Let $t = f_i(t_1, \ldots, t_n), s = f_j(s_1, \ldots, s_m)$, and $i > j$. If f_i is special then $t{\downarrow_K} = \mathrm{i}(t_1{\downarrow_K})$ and $s{\downarrow_K} = p_j \circ (\ldots)$ or $s{\downarrow_K} = p_j$; thus $\mathrm{top}(t{\downarrow_K}) = \mathrm{i} \triangleright \mathrm{top}(s{\downarrow_K})$ implies $t{\downarrow_K} \triangleright_{\mathrm{KB}} s{\downarrow_K}$. Else, $t{\downarrow_K} = p_i \circ (\ldots)$ or $t{\downarrow_K} = p_i$, and $s{\downarrow_K} = p_j \circ (\ldots)$ or $s{\downarrow_K} = p_j$. Since $\mathrm{w}(t{\downarrow_K}) = \mathrm{w}(s{\downarrow_K})$ and $\mathrm{w}(p_i) = \mathrm{w}(p_j)$ we have in fact $t{\downarrow_K} = p_i \circ (\ldots)$ and $s{\downarrow_K} = p_j \circ (\ldots)$, or $t{\downarrow_K} = p_i$ and $s{\downarrow_K} = p_j$. Thus $p_i \triangleright_{\mathrm{KB}} p_j$ implies $t{\downarrow_K} \triangleright_{\mathrm{KB}} s{\downarrow_K}$ in both cases (in the first one due to lexicographic comparison).

(2b) Let $t = f_i(t_1, \ldots, t_n), s = f_i(s_1, \ldots, s_n)$, and for some k, $1 \leqslant k \leqslant n$, let $t_k \succ_{\mathrm{KB}} s_k$ and $\forall l < k : t_l = s_l$. If f_i is special then $t{\downarrow_K} = \mathrm{i}(t_1{\downarrow_K})$ and $s{\downarrow_K} = \mathrm{i}(s_1{\downarrow_K})$; since $t_1 \succ_{\mathrm{KB}} s_1$ the induction hypothesis yields $t_1{\downarrow_K} \triangleright_{\mathrm{KB}} s_1{\downarrow_K}$, hence $t{\downarrow_K} \triangleright_{\mathrm{KB}} s{\downarrow_K}$. Else, $t{\downarrow_K} = p_i \circ (t_1{\downarrow_K} \circ (\cdots \circ (t_n{\downarrow_K} \circ\ w_i) \cdots))$ and $s{\downarrow_K} = p_i \circ (s_1{\downarrow_K} \circ (\cdots \circ (s_n{\downarrow_K} \circ\ w_i) \cdots))$, or $t{\downarrow_K} = p_i \circ (t_1{\downarrow_K} \circ\ t_2{\downarrow_K})$ and $s{\downarrow_K} = p_i \circ\ (s_1{\downarrow_K} \circ\ s_2{\downarrow_K})$. By induction hypothesis, $\forall l < k : t_l{\downarrow_K} = s_l{\downarrow_K}$ and $t_k{\downarrow_K} \triangleright_{\mathrm{KB}} s_k{\downarrow_K}$, thus again in both cases $t{\downarrow_K} \triangleright_{\mathrm{KB}} s{\downarrow_K}$. □

Remark. Lemma 4.2 does not hold for nonground terms in general. Let f_2 be special, let f_1 be a constant of weight 1. Then $f_2(x) \succ_{\mathrm{KB}} f_1$, but $f_2(x){\downarrow_K} = \mathrm{i}(x) \not\triangleright_{\mathrm{KB}} p_1 = f_1{\downarrow_K}$ since $\mathrm{w}(\mathrm{i}(x)) = 1$ and $\mathrm{w}(p_1) \geqslant 2$. This disproves Theorem 3.2.1 in [13] (C. Lautemann, personal communication).

5. Lower bounds

Rewrite systems over monadic signatures with a Knuth–Bendix termination proof have a single exponential upper bound on their derivational complexity (Proposition 6.2). The same is true for rewrite systems where no rewrite step increases the number of special symbols (Proposition 6.1). Proposition 5.1 states that these upper bounds are optimal. For the unrestricted case, by our main theorem an upper bound can always be found in the hierarchy of multiple recursive functions. And indeed, the system given in the proof of Proposition 5.2 has a derivational complexity that is not primitive recursive.

Proposition 5.1 [13]. *There is a finite rewrite system R over a monadic signature such that termination of R is provable using a Knuth–Bendix ordering without special symbols, and* $\mathrm{dc}_R(n) = 2^{\Omega(n)}$.

Proof. Let $\Sigma = \{a, b, c\}$ where a and b are unary and c is a constant, and let R be the system with the two rules

$$a(x) \to b(x), \quad a(b(b(x))) \to b(a(a(x))).$$

Termination of R is established by choosing $\mathrm{w}(a) = \mathrm{w}(b) = 1$ and $a \succ b$. For the lower bound result one easily proves the following claim: For each $n > 0$ there is $m \geqslant \mathrm{Fib}(n)$ (the nth Fibonacci number) so that

$$a^n c \to_R^m b a^{n-1} c.$$

As a consequence, $\mathrm{dh}_R(a^n c) = 2^{\Omega(n)}$; hence $\mathrm{dc}_R(n) = 2^{\Omega(n)}$. Indeed, the claim is trivially true for $n \leqslant 2$, so let $n \geqslant 3$. Then

$$a^n c \to_R^p aba^{n-2}c \to_R^q abba^{n-3}c \to_R ba^{n-1}c,$$

where $p \geqslant \mathrm{Fib}(n-1)$ and $q \geqslant \mathrm{Fib}(n-2)$ by induction hypothesis; thus $p+q+1 > \mathrm{Fib}(n-1)+\mathrm{Fib}(n-2) = \mathrm{Fib}(n)$. □

Proposition 5.2 [13]. *There is a finite rewrite system R such that termination of R is provable using a Knuth–Bendix ordering and without a primitive recursive upper bound on* dc_R.

Proof. Let R over signature Γ from Example 3.1 consist of the three rules

$$\mathrm{i}(x) \circ (y \circ z) \to x \circ (\mathrm{i}(\mathrm{i}(y)) \circ z),$$

$$\mathrm{i}(x) \circ (y \circ (z \circ w)) \to x \circ (z \circ (y \circ w)),$$

$$\mathrm{i}(x) \to x,$$

where $x, y, z, w \in X$. This system is reducing under $\rhd_{\mathrm{KB}}$. For a proof that dc_R denominates the Ackermann function see [10,13]. Interestingly, this example has recently been modified in order to show that simplification orderings do not impose a primitive recursive upper bound on the derivational complexity of string rewrite systems [26]. □

6. Upper bounds

Let R be a finite rewrite system over Σ and let $t_0 \to_R t_1 \to_R \ldots \to_R t_n$ be an R-derivation on ground terms. If termination of R is provable by some Knuth–Bendix ordering $\succ_{\mathrm{KB}}$ then

$$t_0 \succ_{\mathrm{KB}} t_1 \succ_{\mathrm{KB}} \cdots \succ_{\mathrm{KB}} t_n;$$

therefore, by Lemma 4.2, there is a sequence $s_0 = t_0{\downarrow_K}, \ldots, s_n = t_n{\downarrow_K}$ of terms over signature Γ such that

$$s_0 \rhd_{\mathrm{KB}} s_1 \rhd_{\mathrm{KB}} \cdots \rhd_{\mathrm{KB}} s_n.$$

In particular, $\mathrm{w}(s_{j+1}) \leqslant \mathrm{w}(s_j)$. Additionally, since R is finite we know that the size of terms in the sequence is at most linearly growing. More precisely, there is a natural number k such that $t \to_R t'$ implies $|t'|_f \leqslant |t|_f + k$ for $f \in \Sigma$. This is due to the fact that Knuth–Bendix orderings do not allow $|\ell|_x < |r|_x$ for rules $\ell \to r$ and variables x. Hence, by Lemma 4.1 (2),

$$|s_j|_{\mathrm{i}} \leqslant |s_0|_{\mathrm{i}} + j \cdot k.$$

These considerations motivate the following definition.

Definition 6.1. A sequence $s_0, \ldots, s_n$ of Γ-terms is *k-bounded* for $k \in \mathbb{N}$ if $s_0 \rhd_{\mathrm{KB}} \ldots \rhd_{\mathrm{KB}} s_n$ and $|s_j|_{\mathrm{i}} \leqslant |s_0|_{\mathrm{i}} + j \cdot k$. for $j \leqslant n$. We refer to n as the *length* of the sequence.

Since R-derivations translate to k-bounded sequences of Γ-terms via $\downarrow_K$, upper bounds on the length of k-bounded sequences are also upper bounds on the length of R-derivations. By Lemma 4.1 (3), there is a constant $c \in \mathbb{N}$ such that

$$\mathrm{dc}_R(n) \leqslant \max\{m \mid \text{there is a } k\text{-bounded sequence } s_0, \ldots, s_m \text{ with } |s_0| \leqslant c \cdot n\}.$$

Note that the maximal length of all k-bounded sequences starting with a given term is well defined because for any k-bounded sequence $s_0, \ldots, s_i$ there are only finitely many terms s_{i+1} such that $s_0, \ldots, s_i, s_{i+1}$ is k-bounded. Note also that the last term in a maximal sequence is always e.

For a first upper bound result assume that the application of rewrite rules from R cannot increase the number of special symbols. Then the corresponding Γ-sequences are k-bounded with $k = 0$. Here, a single exponential upper bound for dc_R is guaranteed. This, of course, includes the case where we can find a termination proof that completely avoids introducing a special symbol.

Proposition 6.1 [13]. *Let R be a finite rewrite system, and suppose that termination of R is provable using a Knuth–Bendix ordering such that, if a special symbol f exists, then $|\ell|_f \geqslant |r|_f$ for each rule $\ell \to r$ in R. Then* $\mathrm{dc}_R(n) = 2^{O(n)}$.

Proof. Consider a 0-bounded sequence $s_0, \ldots, s_n$ of Γ-terms. Then $|s_j| \leqslant |s_0|$ for $j \leqslant n$ since $|s_j| = |s_j|_\mathrm{i} + |s_j|_\mathrm{o} + |s_j|_\mathrm{e} = |s_j|_\mathrm{i} + 2w(s_j) - 1 \leqslant |s_0|_\mathrm{i} + 2w(s_0) - 1 = |s_0|$. There are $2^{O(n)}$ terms of size $\leqslant n$ over a fixed signature (counting, e.g., the number of strings of length $\leqslant n$ over alphabet Γ, denoting terms in Polish notation). Hence, $\mathrm{dc}_\mathrm{R}(n) = 2^{O(n)}$ by the above remark. □

Another upper bound result is available for systems over monadic signatures. For termination proofs via monotone interpretations, and how they imply upper bounds on derivation lengths, see [10,12,13,21,31] among others.

Proposition 6.2 [13]. *Let R be a finite rewrite system over a monadic signature, and suppose that termination of R is provable using a Knuth–Bendix ordering. Then termination of R is also provable by a linear monotone interpretation over $\mathbb{N}$. Hence,* $\mathrm{dc}_R(n) = 2^{O(n)}$.

Example 6.1. Metivier [22] treats one rule string rewrite systems $R = \{u \to v\}$ where u and v have the same length. For such systems a single exponential upper bound for dc_R can be obtained by Proposition 6.2 as follows. We can assume that the underlying signature has only two unary symbols a and b, and $u = au'$, $v = bv'$ for strings u' and v' [22]. Choosing $\mathrm{w}(a) = \mathrm{w}(b)$ and $a \succ b$ we get $ux \succ_{\mathrm{KB}} vx$; hence Proposition 6.2 as well as Proposition 6.1 are applicable. (In fact, Metivier proves the polynomial upper bound $\mathrm{dc}_R(n) \leqslant n^{|u|}$ and conjectures $\mathrm{dc}_R(n) \leqslant n^2/4$; this was later confirmed by Bertrand [4].)

The basic observation in order to obtain a uniform upper bound for the general case is the uniqueness of k-bounded sequences of maximal length, starting with a given term. This key result will finally lead to recursion equations for calculating the length of such sequences. For $n \in \mathbb{N}$ and $t \neq \mathrm{e}$ define a predecessor function modulo $\triangleright_{\mathrm{KB}}$ by

$$\mathrm{pred}_n(t) = \max_{\triangleright_{\mathrm{KB}}}\{s \in \mathcal{T}_\Gamma \mid t \triangleright_{\mathrm{KB}} s \text{ and } |s|_\mathrm{i} \leqslant n\}.$$

Note that the maximum of this finite set always exists since $\triangleright_{\mathrm{KB}}$ is total on $\mathcal{T}_\Gamma$.

Lemma 6.1. *Among all k-bounded sequences starting with a given term $s_0 \in \mathcal{T}_\Gamma$ there is only one of maximal length, say $s_0, \ldots, s_n$, and for $i < n$ we have $s_{i+1} = \mathrm{pred}_{|s_0|_\mathrm{i}+(i+1)\cdot k}(s_i)$.*

Proof. Suppose $s_0, \ldots, s_i, s', \ldots$ is a k-bounded sequence of maximal length with $s' \neq p$ where $p = \mathrm{pred}_{|s_0|_\mathrm{i}+(i+1)\cdot k}(s_i)$. From $s_i \triangleright_{\mathrm{KB}} s' \neq p$ we deduce $s_i \triangleright_{\mathrm{KB}} p \triangleright_{\mathrm{KB}} s'$. Then, because of $|s'|_\mathrm{i} \leqslant |s_0|_\mathrm{i} + (i+1)\cdot k \leqslant |s_0|_\mathrm{i} + (i+2)\cdot k$, the longer sequence $s_0, \ldots, s_i, p, s', \ldots$ would also be k-bounded, contradicting the assumption. $\square$

For $t \in \mathcal{T}_\Gamma$ and $n \in \mathbb{N}$ define terms $L_n(t)$ and $R_n(t)$ (left and right combs of length n, respectively, ending in t) by

$$L_0(t) = R_0(t) = t, \quad L_{n+1}(t) = L_n(t) \circ \ \mathrm{e}, \quad \mathrm{R}_{n+1}(t) = \mathrm{e} \ \circ R_n(t).$$

We will also simply write L_n and R_n rather than $L_n(\mathrm{e})$ and $R_n(\mathrm{e})$, respectively. The size of L_n and R_n is $2n+1$ and their weight is $n+1$, and it is not difficult to see that L_n is the maximal term of size $2n+1$ and R_n is the minimal term of weight $n+1$ modulo $\triangleright_{\mathrm{KB}}$.

Immediate consequences of the definition of $\triangleright_{\mathrm{KB}}$ are the following five equalities, whose verification is straightforward but tedious, cf. [10, Chap. 5.3].

Lemma 6.2. *Let $t \in \mathcal{T}_\Gamma$ and m, n, $q \in \mathbb{N}$, and define $R = \{R_n \,|\, n \geqslant 0\}$.*
1. $\mathrm{pred}_m(R_{n+1}) = \mathrm{i}^m L_n$.
2. $\mathrm{pred}_m(R_n(t)) = R_n(\mathrm{pred}_m(t))$ *for* $t \notin R$.
3. $\mathrm{pred}_m(\mathrm{i}^{q+1}\mathrm{e}) = \mathrm{i}^{\min\{m,q\}}\mathrm{e}$.
4. $\mathrm{pred}_m(R_n(\mathrm{i}^{q+1}\mathrm{e})) = R_n(\mathrm{i}^{\min\{m,q\}}\mathrm{e})$.
5. $\mathrm{pred}_m(\mathrm{i}R_{n+1}) = \mathrm{i}^m L_n \circ \mathrm{e}$.
6. *if* $m \geqslant q$ *then* $\mathrm{pred}_m(\mathrm{i}^{q+1}\mathrm{e} \circ R_n) = \mathrm{i}^q\mathrm{e} \circ \ \mathrm{i}^{m-q}L_n$.

Lemma 6.3. *Let $s_0, \ldots, s_n$ be the k-bounded sequence starting with s_0 of maximal length. Then there is an index $i \leqslant n$ such that for each index j with $0 \leqslant j \leqslant b$, $b = |s_0|_\mathrm{i} + i\cdot k$, we have $i + j \leqslant n$ and $s_{i+j} = R_{\mathrm{w}(s_0)-1}(\mathrm{i}^{b-j}\mathrm{e})$.*

Proof. Let s_i be the first term of the form $R_{\mathrm{w}(s_0)-1}(\mathrm{i}^m\mathrm{e})$ in the given sequence. Such a term exists since $R_{\mathrm{w}(s_0)-1}$ occurs somewhere in the sequence. (Suppose that this is not the case. Since $\triangleright_{\mathrm{KB}}$ is total, $s_q \triangleright_{\mathrm{KB}} R_{\mathrm{w}(s_0)-1} \triangleright_{\mathrm{KB}} s_{q+1}$ for some $q < n$. Then $s_0, \ldots, s_q, R_{\mathrm{w}(s_0)-1}, s_{q+1}, \ldots, s_n$ would be a k-bounded sequence of length $n+1$, a contradiction.) Now, minimality of i implies $m = b$ by Lemma 6.1 and the fact that $s \triangleright_{\mathrm{KB}} R_p(\mathrm{i}^q\mathrm{e})$ implies $s \trianglerighteq_{\mathrm{KB}} R_p(\mathrm{i}^{q+1}\mathrm{e})$ for $s \in \mathcal{T}_\Gamma$. We conclude by Lemmas 6.1 and 6.2 (4). $\square$

For $s \in \mathcal{T}_\Gamma$ and $k \in \mathbb{N}$ let $d_k(s)$ denote the length of the maximal k-bounded sequence starting with s and cut off after $R_{\mathrm{w}(s)-1}$, that is, the maximal k-bounded sequence where no weight decreasing step occurs. Similarly, let $d'_k(s)$ be the length of the maximal k-bounded sequence starting with $s_0 = s$ and cut off after the first term of the form $s_n = R_{\mathrm{w}(s)-1}(\mathrm{i}^{|s|_\mathrm{i}+n\cdot k}\mathrm{e})$. Both functions are well defined by Lemma 6.3. Note that $d_k(s) = d'_k(s) + |s|_\mathrm{i} + d'_k(s)\cdot k$ as an immediate consequence of Lemma 6.3; thus

$$d'_k = \frac{d_k(s) - |s|_\mathrm{i}}{k+1}. \tag{$*$}$$

Further, let $D_k(m,n)$ be the length of the maximal k-bounded sequence starting with $\mathrm{i}^m L_n$; this is well defined by Lemma 6.1. We will need the fact that the function $\lambda k, m, n.D_k(m,n)$ is monotone in all three arguments. It will be shown that D is in the primitive recursive closure of $\lambda k, m, n.d_k(\mathrm{i}^m L_n)$ in Lemma 6.4; therefore we concentrate on computing d in Lemma 6.5 and finally relate d to an appropriate number theoretic function in Lemma 6.6.

Lemma 6.4. *For* $m, n, k \in \mathbb{N}$,

$$D_k(m,0) = m,$$

$$D_k(m,n+1) = d_k(\mathrm{i}^m L_{n+1}) + 1 + D_k(m + (d_k(\mathrm{i}^m L_{n+1}) + 1) \cdot k, n).$$

Proof. The first equality expresses the fact that $\mathrm{i}^m\mathrm{e}, \dots, \mathrm{e}$ is the corresponding maximal k-bounded sequence. Now consider a maximal k-bounded sequence starting with $\mathrm{i}^m L_{n+1}$. Its prefix including R_{n+1} has length $d_k(\mathrm{i}^m L_{n+1})$, hence the second equality by Lemmas 6.1 and 6.2 (1). □

Lemma 6.5. *For* $s \in \mathcal{T}_\Gamma$, $t \in \mathcal{T}_\Gamma$ *with* $\mathrm{w}(t) > 1$, *and* $m, n, k \in \mathbb{N}$,

$$d_k(R_n(s)) = d_k(s), \tag{1}$$

$$d_k(\mathrm{i}^m\mathrm{e}) = m, \tag{2}$$

$$d_k(\mathrm{i}t) = d_k(t) + 1 + d_k(\mathrm{i}^{|t|_\mathrm{i}+1+(d_k(t)+1)k} L_{\mathrm{w}(t)-2} \circ \mathrm{e}), \tag{3}$$

$$d_k(\mathrm{i}^{m+1}\mathrm{e} \circ s) = d_k(s) + 1 + d_k(\mathrm{i}^m\mathrm{e} \circ \mathrm{i}^{|s|_\mathrm{i}+1+(d_k(s)+1)k} L_{\mathrm{w}(s)-1}), \tag{4}$$

$$d_k(R_{n+1}(\mathrm{i}^m\mathrm{e}) \circ s) = d_1 + d_k(\mathrm{i}^{m+|s|_\mathrm{i}+d_1 k} L_n \circ L_{\mathrm{w}(s)}), \textit{where } d_1 = d_k(\mathrm{i}^m(\mathrm{e} \circ s) + 1, \tag{5}$$

$$\begin{gathered} d_k(t \circ L_n) = \begin{cases} c + d_k(\mathrm{i}^{|t|_\mathrm{i}+ck} L_{\mathrm{w}(t)-2} \circ L_{n+1}) & \textit{if } |t|_\mathrm{i} + abk \geqslant 1, \\ a(b+1) + d_k(\mathrm{i}^{ak} L_{\mathrm{w}(t)-2} \circ L_{n+1}) & \textit{else}, \end{cases} \\ \textit{where } a = d_k(L_n) + 1,\ b = d'_{ak}(t),\ c = a(b+1) + d_2, \\ d_2 = d_k(\mathrm{i}^{|t|_\mathrm{i}+abk-1}\mathrm{e} \circ \mathrm{i}^{ak+1} L_n) + 1. \end{gathered} \tag{6}$$

Proof. These equalities follow from Lemmas 6.1 and 6.2: (1) by Lemmas 6.2 (1) and 6.2 (2), (2) by 6.2 (3), (3) by 6.2 (5), (4) by 6.2 (6). The proof of (5) is similar. For the proof of (6), if $|t|_\mathrm{i} + abk \geqslant 1$ then

$$\begin{aligned} d_k(t \circ L_n) &= ab + a - 1 + d_k(R_{\mathrm{w}(t)-1}(\mathrm{i}^{|t|_\mathrm{i}+abk}\mathrm{e}) \circ R_n) \\ &= a(b+1) + d_k(R_{\mathrm{w}(t)-1}(\mathrm{i}^{|t|_\mathrm{i}+abk-1}\mathrm{e}) \circ \mathrm{i}^{ak+1} L_n) \\ &= a(b+1) + d_2 + d_k(\mathrm{i}^{|t|_\mathrm{i}+(a(b+1)+d_2)k} L_{\mathrm{w}(t)-2} \circ L_{n+1}), \end{aligned}$$

where $d_2 = d_k(\mathrm{i}^{|t|_\mathrm{i}+abk-1}\mathrm{e} \circ \mathrm{i}^{ak+1}L_n) + 1$ by using (5) for the last equality. Otherwise we have $|t|_\mathrm{i} + abk = 0$; that is, $|t|_\mathrm{i} = 0$ and $bk = 0$. Then $d_k(t \circ L_n) = a(b+1) - 1 + d_k(R_{\mathrm{w}(t)-1} \circ R_n) = a(b+1) + d_k(\mathrm{i}^{ak}L_{\mathrm{w}(t)-2} \circ L_{n+1})$. $\square$

Define $P : \mathbb{N}^5 \to \mathbb{N}$ by $P_k(m_1, n_1, m_2, n_2) = d_k(\mathrm{i}^{m_1}L_{n_1} \circ \mathrm{i}^{m_2}L_{n_2})$.

Lemma 6.6. *For $m, n, m_i, n_i, k \in \mathbb{N}$,*

$$P_k(0, 0, m, 0) = m, \tag{7}$$

$$P_k(0, 0, 0, n+1) = P_k(0, n, 0, 0), \tag{8}$$

$$\begin{aligned} P_k(0, 0, m+1, n+1) = {} & P_k(0, 0, m, n+1) + 1 \\ & + P_k(m + 1 + (P_k(0, 0, m, n+1) + 1)k, n, 0, 0), \end{aligned} \tag{9}$$

$$P_k(m_1 + 1, 0, m_2, n) = P_k(0, 0, m_2, n) + 1 + P_k(m_1, 0, m_2 + 1 + (P_k(0, 0, m_2, n) + 1)k, n), \tag{10}$$

$$P_0(0, n_1 + 1, 0, n_2) = (P_0(0, 0, 0, n_2) + 1)(P_0(0, n_1, 0, 0) + 1) + P_0(0, n_1, 0, n_2 + 1), \tag{11}$$

$$P_{k+1}(0, 1, 0, 0) = 1 + P_{k+1}(k + 1, 0, 0, 1), \tag{12}$$

$$\begin{aligned} & P_{k+1}(0, n+2, 0, 0) = C + P_{k+1}(C(k+1), n+1, 0, 1) \\ & \qquad \textit{where } B = P_{k+1}(0, 0, 0, n+2)/(k+2), \\ & \qquad\qquad C = B + 1 + D, \\ & \qquad\qquad D = P_{k+1}(B(k+1) - 1, 0, k+2, 0) + 1, \end{aligned} \tag{13}$$

$$\begin{aligned} & P_k(m+1, n_1+1, 0, n_2) = C + P_k(m + 1 + Ck, n_1, 0, n_2 + 1) \\ & \qquad \textit{where } A = P_k(0, 0, 0, n_2) + 1, \\ & \qquad\qquad B = (P_{Ak}(0, 0, m+1, n_1+1) - m - 1)/(k+1), \\ & \qquad\qquad C = A(B+1) + D, \\ & \qquad\qquad D = P_k(m + ABk, 0, Ak + 1, n_2) + 1. \end{aligned} \tag{14}$$

Proof. All equalities are immediate consequences of Lemma 6.5. For the first three note that $d_k(\mathrm{e} \circ s) = d_k(R_1(s)) = d_k(s)$ by (1). Then we have (7) by (2), (8) from $L_{n+1} = L_n \circ \mathrm{e}$, (9) by (3), (10) by (4), and (11)–(14) by (6) using (*) for the computation of B. $\square$

The equations in Lemma 6.6 are (part of) the definition of a multiple recursive function. To see this, note first that each instance of a recursive call of P is covered by some left-hand side. Then add an equation for the missing cases (for instance, $P_k(m_1, n_1 + 1, m_2 + 1, n_2) = 0$), and use the transformation

$$P_z(x_1, y_1, x_2, y_2) \mapsto Q(y_1 + y_2, y_1, x_1, x_2, z).$$

After eliminating P, the resulting equational system has the following property. Let $Q(t_1, \ldots, t_5)$ be a left-hand side and let $Q(s_1, \ldots, s_5)$ be an arbitrary subterm of the corresponding right-hand side. Then, after evaluating terms to numbers, the list $(t_1, \ldots, t_5)$ is greater than the list $(s_1, \ldots, s_5)$ when compared lexicographically from left to right. Thus Q, and therefore P, is a multiple recursive function [24]. Note that it even suffices to compare $(t_1, \ldots, t_4)$ with $(s_1, \ldots, s_4)$; hence P is 4-recursive.

We have already shown in Lemma 6.4 that $\lambda k, m, n.D_k(m, n)$ is primitive recursive in $\lambda k, m, n.d_k(\mathrm{i}^m L_n)$, so we deduce from

$$d_k(\mathrm{i}^m L_n) = d_k(\mathrm{e} \circ \mathrm{i}^m L_n) = d_k(\mathrm{i}^0 L_0 \circ \mathrm{i}^m L_n) = P_k(0, 0, m, n)$$

that also d and D are 4-recursive. By the remark after Definition 6.1 and since $L_n \trianglerighteq_{\mathrm{KB}} s$ for all terms s of size n, we know that

$$\mathrm{dc}_R(n) \leqslant D_k(0, c \cdot n) \leqslant D_{\max\{c,k\}}(0, \max\{c, k\} \cdot n)$$

for the above defined constants c and k, which solely depend on R. Defining $b : \mathbb{N}^2 \to \mathbb{N}$ by

$$b(r, n) = D_r(0, r \cdot n),$$

we finally arrive at our main result.

Theorem. *There is a multiple recursive function $b : \mathbb{N}^2 \to \mathbb{N}$ such that for all finite rewrite systems R that have a termination proof using a Knuth–Bendix ordering there exists $r \in \mathbb{N}$ with $\mathrm{dc}_R(n) < b(r, n)$.*

The proof of Proposition 5.2 shows that 2-recursive upper bounds are the best we can expect for unrestricted Knuth–Bendix termination proofs. Since we could only prove the existence of a 4-recursive upper bound, the question whether 2- or 3-recursive upper bounds (3- or 4-recursive lower bounds, respectively) exist was left as an open problem. Recently, Lepper [18] solved this in establishing a uniform 2-recursive upper bound for rewrite systems R where $R \subseteq \succ_{\mathrm{KB}}$ for some (generalized) Knuth–Bendix ordering $\succ_{\mathrm{KB}}$. (Note that the results in Section 6 also hold under the slightly weaker assumption that R is terminating via some "ground" Knuth–Bendix ordering, that is, if $t \to_R s$ implies $t \succ_{\mathrm{KB}} s$ for ground terms t and s.) This is complemented by [19] where it has been shown that the class of functions computable by such rewrite systems coincides with the class of functions computable in linear Ackermann time on register machines; see also Bonfante [5] for related results on restricted forms of Knuth–Bendix orderings.

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